Spontaneous Symmetry Breaking in General Relativity. Brane World Concept.

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(Dated: October 16, 2009)

Abstract

Gravitational properties of a hedge-hog type topological defect in two extra dimensions are considered in General Relativity employing a vector as the order parameter. All previous considerations were done using the order parameter in the form of a multiplet in a target space of scalar fields. The difference of these two approaches is analyzed and demonstrated in detail. Regular solutions of the Einstein equations are studied analytically and numerically. It is shown that the existence of a negative cosmological constant is sufficient for the spontaneous symmetry breaking of the initially plain bulk. Regular configurations have a growing gravitational potential and are able to trap the matter on the brane. If the energy of spontaneous symmetry breaking is high, the gravitational potential has several points of minimum. Identical in the uniform bulk spin-less particles, being trapped within separate minima, acquire different masses and appear to the observer on brane as different particles with integer spins.

PACS numbers: 04.50.+h, 98.80.Cq

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I. INTRODUCTION

The theories of brane world and multidimensional gravity are widely discussed in the literature. A natural physical concept is that a distinguished surface in the space-time manifold is a topological defect appeared as a result of a phase transition with spontaneous symmetry breaking. The macroscopic theory of phase transitions allows to consider the brane world concept self-consistently, even without the knowledge of the nature of physical vacuum. The properties of topological defects (strings, monopoles, ...) are generally described with the aid of multiplet of scalar fields forming a hedgehog configuration in extra dimensions (see [1] and references there in). The scalar multiplet plays the role of the order parameter. The hedgehog configuration forms a vector proportional to a unit vector in the Euclidean target space of scalar fields. This model is self-consistent, but it is not the only way for generalization of a plane monopole to the curved space-time.

In a flat space-time there is no difference between a vector and a hedgehog-type multiplet of scalar fields. On the contrary, in curved space-time scalar multiplets and vectors are transformed differently. For this reason in general relativity the two approaches (a multiplet of scalar fields and a vector order parameter) give different results which are worth to be compared. It looks more difficult to deal with a vector order parameter, and, probably, it is the reason why I couldn't find in the literature any papers considering phase transitions with a hedgehog-type vector order parameter in general relativity.

II. GENERAL FORMULAE

A. Lagrangian

The order parameter enters the Lagrangian via scalar bilinear combinations of its derivatives and via a scalar potential V allowing the spontaneous symmetry breaking. If ϕ_I is a vector order parameter, then V should be a function of the scalar $\phi^K\phi_K = g^{IK}\phi_I\phi_K$, and a bilinear combination of the derivatives is a tensor

$$S_{IKLM} = \phi_{I;K}\phi_{L;M}. \tag{1}$$

Index $_{;K}$ is used as usual for covariant derivatives. There are three ways to simplify S_{IKLM} into scalars, so the most general form of the scalar S, formed via contractions of S_{IKLM}, is

$$S = A\left(\phi^K_{;K}\right)^2 + B\phi^L_{;K}\phi^{;K}_L + C\phi^M_{;K}\phi^K_{;M}, \tag{2}$$

where A, B, and C are arbitrary constants. Different topological defects can be classified by these parameters. In curved space-time the scalar S depends not only on the derivatives of the order parameter, but also on the derivatives of the metric tensor. This is the principle difference between a vector and a multiplet of scalar fields.

The general form of the Lagrangian determining gravitational properties of topological defects with a vector order parameter is

$$L\left(\phi_I, g^{IK}, \frac{\partial g_{IK}}{\partial x^L}\right) = L_g + L_d, \tag{3}$$

where

$$L_g = \frac{R}{2\kappa^2}, \tag{4}$$

$$L_d = A\left(\phi^K_{;K}\right)^2 + B\phi^I_{;K}\phi^{;K}_I + C\phi^I_{;K}\phi^K_{;I} - V\left(\phi^K\phi_K\right). \tag{5}$$

L_g is the Lagrangian of the gravitational field, R is the scalar curvature of space-time, κ^2 is the (multidimensional) gravitational constant, and L_d is the Lagrangian of a topological defect. Covariant derivation

$$\phi_{P;M} = \frac{\partial\phi_P}{\partial x^M} - \frac{1}{2}g^{LA}\left(\frac{\partial g_{AM}}{\partial x^P} + \frac{\partial g_{AP}}{\partial x^M} - \frac{\partial g_{MP}}{\partial x^A}\right)\phi_L \tag{6}$$

and razing of indexes $\phi^K = g^{IK}\phi_I$ contain g^{IK} and $\frac{\partial g_{IK}}{\partial x^L}$, and for this reason it is convenient to express the Lagrangian as a function of ϕ_I, g^{IK}, and $\frac{\partial g_{IK}}{\partial x^L}$.

B. Energy-momentum tensor

Varying the Lagrangian L_d (5) with respect to δg^{IK} and having in mind that

$$\delta g_{IK} = -g_{KM}g_{IN}\delta g^{NM}, \tag{7}$$

we get the following expression for the energy-momentum tensor [4]:

$$T_{IK} = \tfrac{2}{\sqrt{-g}}\left[\tfrac{\partial\sqrt{-g}L_d}{\partial g^{IK}} + g_{QK}g_{PI}\tfrac{\partial}{\partial x^L}\left(\sqrt{-g}\tfrac{\partial L_d}{\partial\frac{\partial g_{PQ}}{\partial x^L}}\right)\right] \tag{8}$$

In the case of the vector order parameter the potential $V\left(\phi^K\phi_K\right) = V\left(g^{IK}\phi_I\phi_K\right)$ also undergoes the variation with respect to δg^{IK}.

It is worth to conduct further derivations with account of specific properties of particular topological defects.

III. GLOBAL STRING IN EXTRA DIMENSIONS

In my previous papers with Bronnikov (see [1] and references there in) we considered global monopoles and strings as topological defects with the order parameter in the form of a hedge-hock type multiplet of scalar fields in some flat target space. The aim of this paper is to describe these defects using vector order parameter and compare the results.

A. Metric

The direction of the vector specifies one coordinate, and in the most simple case the system is uniform and isotropic with respect to all other coordinates. In our recent paper [1] we presented the detailed properties of global strings in two extra dimensions. For this reason I consider below a topological defect in the space-time with two extra dimensions. The order parameter is a space-like vector ($g^{IK}\phi_I\phi_K < 0$) directed normally from the brane hypersurface and depending on the only one specific coordinate, namely – the distance from the brane. The whole ($D = d_0 + 2$)-dimensional space-time has the structure $\mathrm{M}^{d_0}\times \mathrm{R}^1\times \Phi^1$ and the metric

$$ds^2 = e^{2\gamma(l)}\eta_{\mu\nu}dx^\mu dx^\nu - \left(dl^2 + e^{2\beta(l)}d\varphi^2\right), \tag{9}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, ..., -1)$ is the d_0-dimensional Minkovsky brane metric ($d_0 > 1$), and φ is the angular cylindrical coordinate in extra dimensions. γ and β are functions of the distinguished extradimensional coordinate l – the distance from the center, i.e. from the brane. $e^{\beta(l)} = r\,(l)$ is the circular radius. Greek indices $\mu, \nu, ..$ correspond to d_0-dimensional space-time on the brane, and $I, K, ...$ – to all $D = d_0 + 2$ coordinates. The metric tensor g_{IK} is diagonal, and its nonzero components are denoted as follows:

$$g_{IK} = \begin{cases} e^{2\gamma}, & I = K = 0, \\ -e^{2\gamma}, & 0 < I = K < d_0, \\ -1, & I = K = d_0, \\ -e^{2\beta}, & I = K = \varphi. \end{cases} \tag{10}$$

The curvature of the metric on brane due to the matter is supposed to be much smaller than the curvature of the bulk caused by the brane formation.

B. Regularity conditions

If the influence of matter on brane is neglected, then there is no physical reason for singularities, and the selfconsistent structure of a topological defect should be regular. A necessary condition of regularity is finiteness of all invariants of the Riemann tensor of curvature. The nonzero components of the Riemann tensor are

$$R^{AB}_{\ \ CD} = \begin{cases} -\gamma'^2 \left(\delta^A_C \delta^B_D - \delta^A_D \delta^B_C\right), & A, B, C, D < d_0, \\ -\beta'\gamma', \quad A = C = \varphi, & B, D < d_0, \\ -\left(\gamma'' + \gamma'^2\right)\delta^B_D, \quad A = C = d_0, & B, D < d_0, \\ -\left(\beta'' + \beta'^2\right), \quad A = C = d_0, & B = D = \varphi. \end{cases} \tag{11}$$

Here prime denotes d/dl. One of the invariants of the Riemann tensor is the Kretchmann scalar $K = R^{AB}_{\ \ CD} R^{CD}_{\ \ AB}$, which is the sum of all nonzero $R^{AB}_{\ \ CD}$ squared. I.e. all the nonzero components of the Riemann tensor, and namely

$$\gamma', \quad \gamma'' + \gamma'^2, \quad \beta'\gamma', \quad \beta'' + \beta'^2 \tag{12}$$

must be finite. $r = 0$ is a singular point of the cylindrical coordinate system. The absence of curvature singularity in the center follows from the last condition (12) . Let

$$\beta'' + \beta'^2 = c < \infty \quad at \quad l = 0. \tag{13}$$

Integrating (13) in the vicinity of the center we have

$$\beta' = \frac{1}{l} + \frac{1}{3} cl + O\left(l^3\right). \tag{14}$$

Relation (14) ensures the correct ($= 2\pi$) circumference-to-radius ratio, or, equivalently, $dr^2 = dl^2$ at $l \to 0$. Finiteness of $\beta'\gamma'$ at $l = 0$ is fulfilled if

$$\gamma' = O\left(l\right) \tag{15}$$

at $l \to 0$, or smaller.

C. Vector order parameter

Our aim is to consider the order parameter as a vector in extra dimensions directed normally from the Minkovsky hypersurface. In the cylindrical coordinate system of extra dimensions the only nonzero component of the vector order parameter is

$$\phi_{d_0} \equiv \phi. \tag{16}$$

In the space-time with the metric (9) the covariant derivative

$$\phi_{I;K} = \delta_I^{d_0} \delta_K^{d_0} \phi' - \frac{1}{2} \delta_{IK} g^{II} g'_{II} \phi \tag{17}$$

is a symmetric tensor: $\phi_{I;K} = \phi_{K;I}$. For this reason $\phi^I_{;K} \phi^{;K}_I = \phi^I_{;K} \phi^K_{;I}$, and the Lagrangian (5) takes the form

$$L_d = A \left(\phi' + \tfrac{1}{2} \phi \sum_K g^{KK} g'_{KK} \right)^2 + \widetilde{B} \left(\phi'^2 + \tfrac{1}{4} \phi^2 \sum_L \left(g^{LL} g'_{LL} \right)^2 \right) - V\left(-\phi^2\right) \tag{18}$$

and contains only two arbitrary constants A and $\widetilde{B} = B + C$. In (18) we set $g^{d_0 d_0} = -1$ in accordance with (9). However one should keep in mind that (18) cannot be used in (8). To derive the energy-momentum tensor (8) one should use the Lagrangian (5), and set $g^{d_0 d_0} = -1$, $\left(g^{d_0 d_0} \right)' = 0$ after differentiation. Nevertheless, the field equation can be derived using (18) in the general formula

$$\frac{1}{\sqrt{-g}} \left(\frac{\partial \sqrt{-g} L_d}{\partial \phi'} \right)' - \frac{\partial L_d}{\partial \phi} = 0. \tag{19}$$

In the space-time with metric (9) the sums in (18) are

$$S_n = \tfrac{1}{2^n} \sum_K \left(g^{KK} g'_{KK} \right)^n = d_0 \gamma'^n + \beta'^n, \quad n = 1, 2, ... \tag{20}$$

and the determinant of the metric tensor is

$$g = (-1)^{D-1} e^{2(d_0 \gamma + \beta)}. \tag{21}$$

D. Field equation

We consider below the case $A \neq 0$, $\widetilde{B} = 0$. The case $A = 0$, $\widetilde{B} \neq 0$ will be considered elsewhere. Substituting (18) with $A = \frac{1}{2}$, $\widetilde{B} = 0$ into (19) we get the following field equation in the case of vector order parameter

$$\left[\phi' + (d_0\gamma' + \beta')\,\phi\right]' + \frac{\partial V}{\partial \phi} = 0. \tag{22}$$

In the case of the multiplet of scalar fields we had [1]:

$$\phi'' + \phi'\,(d_0\gamma' + \beta') - \phi e^{-2\beta} + \frac{\partial V}{\partial \phi} = 0. \tag{23}$$

Unlike (23), the field equation (22) doesn't depend directly on β (and thus on the circular radius $r = \ln\beta$), but instead includes second derivatives of the metric tensor. In the flat space-time $\gamma' = 0$, $\beta' = \frac{1}{l}$, $\beta'' = -\frac{1}{l^2}, e^{-2\beta} = \frac{1}{l^2}$, and both field equations reduce to

$$\phi'' + \frac{1}{l}\phi' - \frac{1}{l^2}\phi + \frac{\partial V}{\partial \phi} = 0. \tag{24}$$

E. Energy-momentum tensor

The energy-momentum tensor (8) inevitably contains second derivatives. However, with the aid of the field equation (22) , the second derivatives can be excluded. The final result of a rather wearing derivation is

$$T_I^K = \tfrac{1}{2}\delta_I^K\left[\phi' + (d_0\gamma' + \beta')\,\phi\right]^2 + \delta_I^K V + \left(\delta_I^{d_0}\delta_{d_0}^K - \delta_I^K\right)\tfrac{\partial V}{\partial \phi}\phi \tag{25}$$

Unlike the scalar multiplet case, the energy-momentum tensor (25) contains not only the potential V, but also its derivative $\frac{\partial V}{\partial \phi}$.

Correctness of (25) is checked by the derivation of the covariant divergence $T^K_{I;K}$ (actually $T^K_{d_0;K}$). Again, with the aid of the field equation (22) we confirm that $T^K_{d_0;K} = 0$.

F. Einstein equations

The same way as in [1] we use the Einstein equations in the form

$$R_I^K = \kappa^2 \widetilde{T}_I^K, \tag{26}$$

where R_I^K is the Ricci tensor,

$$R_I^K = \begin{cases} \delta_I^K\left[\gamma'' + \gamma'\,(d_0\gamma' + \beta')\right], \quad I < d_0 \\ \delta_{d_0}^K\left[d_0\,(\gamma'' + \gamma'^2) + \beta'' + \beta'^2\right], \quad I = d_0 \\ \delta_\varphi^K\left[\beta'' + \beta'\,(d_0\gamma' + \beta')\right], \quad I = \varphi \end{cases} \tag{27}$$

and

$$\widetilde{T}_I^K = T_I^K - \tfrac{1}{d_0}\delta_I^K T = -\tfrac{1}{d_0}\delta_I^K \left[\phi' + \left(d_0\gamma' + \beta'\right)\phi\right]^2 - \delta_I^K \tfrac{2}{d_0} V + \delta_I^K \left(\delta_I^{d_0} + \tfrac{1}{d_0}\right) \tfrac{\partial V}{\partial \phi}\phi.$$

In the case of the vector order parameter the set of Einstein equations

$$\gamma'' + \gamma'\left(d_0\gamma' + \beta'\right) = \kappa^2 \left[-\tfrac{1}{d_0}\left[\phi' + \left(d_0\gamma' + \beta'\right)\phi\right]^2 - \tfrac{2V}{d_0} + \tfrac{1}{d_0}\tfrac{\partial V}{\partial \phi}\phi\right] \tag{28}$$

$$d_0\gamma'' + \beta'' + d_0\gamma'^2 + \beta'^2 = \kappa^2 \left[-\tfrac{1}{d_0}\left[\phi' + \left(d_0\gamma' + \beta'\right)\phi\right]^2 - \tfrac{2V}{d_0} + \left(1 + \tfrac{1}{d_0}\right)\tfrac{\partial V}{\partial \phi}\phi\right] \tag{29}$$

$$\beta'' + \beta'\left(d_0\gamma' + \beta'\right) = \kappa^2 \left[-\tfrac{1}{d_0}\left[\phi' + \left(d_0\gamma' + \beta'\right)\phi\right]^2 - \tfrac{2V}{d_0} + \tfrac{1}{d_0}\tfrac{\partial V}{\partial \phi}\phi\right] \tag{30}$$

consists of three first order equations with respect to γ', β', and ϕ . Both γ and β do not enter the equations $(28-30)$ directly, only via the derivatives. In the case of a scalar multiplet order parameter, see eq.$(14-16)$ in [1], β enters the Einstein equations directly, and the system of equations is of the fourth order.

The field equation (22) is not independent. It is a consequence of the Einstein equations $(28-30)$ due to the Bianchi identity.

1. *First integral*

Excluding the second derivatives γ'' and β'' in the set $(28-30)$, we get the relation

$$\left(d_0\gamma' + \beta'\right)^2 - \left(d_0\gamma'^2 + \beta'^2\right) = -\kappa^2 \left\{\left[\phi' + \left(d_0\gamma' + \beta'\right)\phi\right]^2 + 2V\right\}, \tag{31}$$

which can be considered as a first integral of the system $(28-30)$.

2. *Further simplification*

The equations (28) and (30) have the same right hand sides. Extracting one from the other we get the equation

$$\left(\gamma' - \beta'\right)' + \left(\gamma' - \beta'\right)\left(d_0\gamma' + \beta'\right) = 0, \tag{32}$$

which can be used instead of one of the equations (28) and (30). With the aid of the relations (31) and (32) the complete set of equations can be reduced to a more simple form. Introducing new functions

$$U = \gamma' - \beta', \quad W = d_0\gamma' + \beta', \quad Z = \phi' + W\phi, \tag{33}$$

we get the set of four first order equations

$$U' = -UW \tag{34}$$

$$W' = \kappa^2 \frac{d_0+1}{d_0}\left(\frac{\partial V}{\partial \phi}\phi - 2V - Z^2\right) - W^2 \tag{35}$$

$$\phi' \;=\; Z - W\phi \tag{36}$$

$$Z' \;=\; -\frac{\partial V}{\partial \phi}. \tag{37}$$

Functions β', γ', and their combination $S_2 = d_0\gamma'^2 + \beta'^2$ (20) are expressed via U and W as follows:

$$\gamma' = \frac{U+W}{d_0+1} \qquad \beta' = \frac{W - d_0 U}{d_0+1} \qquad S_2 = \frac{d_0 U^2 + W^2}{d_0+1}. \tag{38}$$

In terms of U, W, and Z the first integral (31) ,

$$W^2 - U^2 = -\kappa^2 \frac{d_0+1}{d_0}\left\{Z^2 + 2V\right\}, \tag{39}$$

allows to simplify (35) even more:

$$W' = \kappa^2 \frac{d_0+1}{d_0}\frac{\partial V}{\partial \phi}\phi - U^2. \tag{40}$$

The set of equations

$$\begin{gathered} U' = -UW \\ W' = \kappa^2 \tfrac{d_0+1}{d_0}\tfrac{\partial V}{\partial \phi}\phi - U^2 \\ \phi' = Z - W\phi \\ Z' = -\tfrac{\partial V}{\partial \phi} \end{gathered} \tag{41}$$

is most convenient for both analytical and numerical analysis.

G. General analysis of equations

Equations $(28-30)$ are invariant against adding arbitrary constants to γ and β. Without loss of generality we can set

$$\gamma(0) = 0. \tag{42}$$

Requirement of regularity in the center dictates the condition (14) , and, if we do not consider configurations with angle deficit (or surplus), we have

$$r = e^{\beta} = l, \quad l \to 0. \tag{43}$$

Integrating (32) with boundary conditions $(42, 43)$ we get

$$\gamma' - \beta' = -e^{-(d_0\gamma+\beta)}. \tag{44}$$

It follows from (44) that $\beta' > \gamma'$ everywhere.

Recall that topological defects, formed as multiplets of scalar fields [1], are of three types. Integral curves can terminate with:

A) infinite circular radius $r(l)$ at $l \to \infty$;

B) finite circular radius $r_\infty = r(\infty) = const < \infty$;

C) second center $r = 0$ at some finite $l = l_c$.

In the vector order parameter case the situation is different. Equation (44) allows to prove that a regular configuration cannot terminate neither with a finite value of circular radius r_∞ at $l \to \infty$, nor in the second center.

Suppose for a moment, that $r_\infty = const < \infty$. Then $\beta'(\infty) = 0$, and (44) reduces to $\gamma' = -\frac{1}{r_\infty} e^{-d_0\gamma}$ at $l \to \infty$. After integration we get

$$e^{d_0\gamma} = \frac{d_0}{r_\infty}(l_0 - l), \tag{45}$$

l_0 is a constant of integration. The l.h.s. is obviously positive, while the r.h.s. becomes negative and infinitely large at $l \to \infty$. Thus $r_\infty = const < \infty$ is impossible.

The second center is also impossible. In the vicinity of the second center the l.h.s. of (44) becomes large positive due to $-\beta'$, and the r.h.s. remains negative.

We come to the conclusion that regular configurations of topological defects with the vector order parameter start at the center $l = 0$ and terminate at $l \to \infty$ with infinitely growing circular radius $r(l) \to \infty$.

It follows from the requirement of regularity (15) that $\gamma' = \gamma_0'' l$ at $l \to 0$. From the first integral (31) we find the relation between γ_0'', ϕ_0', and V_0 :

$$\gamma_0'' = -\frac{\kappa^2}{d_0}\left(2\phi_0'^2 + V_0\right), \tag{46}$$

where V_0 is the value of the potential at the center $l = 0$. In both cases (scalar multiplet and vector order parameter) the value $\phi_0' = \phi'(0)$ is not restricted by the equations. The difference is that in the scalar multiplet case ϕ_0' becomes fixed unanimously by the requirement of regularity, and in the case of vector order parameter ϕ_0' remains a free parameter.

H. Asymptotic behavior

Condition of regularity requires that γ' is finite everywhere. Within the area of regularity it tends to a fixed finite value γ'_∞ at $l \to \infty$. As soon as $r(l) \to \infty$ at $l \to \infty$, we see from (44) that $\gamma' - \beta' \to 0$. Thus $\beta'(\infty) = \gamma'_\infty$. The field $\phi(l)$ also tends to its finite value $\phi_\infty = \phi(\infty)$. Then it follows from the field equation (22) that $\frac{\partial V}{\partial \phi} \to 0$ at $l \to \infty$, i.e. the regular configuration terminates at an extremum of the potential $V(\phi)$. Let $V_\infty = V(\phi_\infty)$, $V'(\phi_\infty) = 0$. From the first integral (31) we find the limiting value γ'_∞ :

$$\gamma'_\infty = \sqrt{-\frac{2\kappa^2 V_\infty}{(d_0+1)\left[d_0 + (d_0+1)\kappa^2\phi_\infty^2\right]}}. \tag{47}$$

A necessary condition of existence of regular configurations of topological defects with the vector order parameter is $V_\infty < 0$.

To find the asymptotic behavior of $\phi(l)$ and $W(l)$ we linearize the equations (41) at $l \to \infty$:

$$\phi = \phi_\infty + \delta\phi, \quad W = (d_0+1)\gamma'_\infty + \delta W, \tag{48}$$

$$\begin{aligned} \delta W' &= \kappa^2 \tfrac{d_0+1}{d_0} V''_\infty \phi_\infty \delta\phi \\ \delta\phi' &= \delta Z - (d_0+1)\gamma'_\infty \delta\phi - \phi_\infty \delta W \\ \delta Z' &= -V''_\infty \delta\phi \end{aligned} \tag{49}$$

Here primes denote derivatives d/dl, $(\delta W' = d\delta W/dl, ...)$, except $V''_\infty = \frac{\partial^2 V}{\partial \phi^2}|_{\phi=\phi_\infty}$. Excluding δZ and δW, we get the second order linear homogeneous equation for $\delta\phi$:

$$\delta\phi'' + (d_0+1)\gamma'_\infty \delta\phi' + \frac{2\kappa^2 |V_\infty| V''_\infty}{d_0 (d_0+1)\gamma'^2_\infty}\delta\phi = 0. \tag{50}$$

If the extremum of the potential is minimum ($V''_\infty > 0$) its nontrivial solution vanishes at $l \to \infty$:

$$\delta\phi = A e^{\lambda_+ l} + B e^{\lambda_- l}, \tag{51}$$

where A and B are constants of integration, and both eigenvalues

$$\lambda_\pm = -\frac{(d_0+1)\gamma'_\infty}{2}\left(1 \mp \sqrt{1 - \frac{8\kappa^2 |V_\infty| V''_\infty}{d_0 (d_0+1)^3 \gamma'^4_\infty}}\right) \tag{52}$$

are either negative, or have negative real parts. Absence of growing solutions is the reason why ϕ'_0 remains a free parameter in the vector order parameter case.

The asymptotic behavior of the field $\phi(l)$ far from the center is determined by two constant parameters of the symmetry breaking potential near its extremum, namely V_∞ and V''_∞. If the extremum is minimum, $V''_\infty > 0$, then the expression under the root can be both positive and negative. So $\phi(l)$ can tend to ϕ_∞ either smoothly, or with oscillations. In the space of physical parameters the boundary between smooth and oscillating solutions is determined by the relation

$$\frac{8\kappa^2 |V_\infty| V''_\infty}{d_0 (d_0+1)^3 \gamma'^4_\infty} = 1. \tag{53}$$

Oscillating behavior of the field $\phi(l)$ induces oscillations of β' and γ'. If γ' changes sign, then $\gamma(l)$ can have minimums. Remind, that γ acts as a gravitational potential, so the matter can be trapped near the minimums of $\gamma(l)$.

Usually $\phi = 0$ is a maximum of the potential $V(\phi)$. It is also an extremum, $\partial V/\partial\phi = 0$ at $\phi = 0$. Regular configurations, starting from the center $l = 0$ with $\phi(0) = 0$, can terminate at $l \to \infty$ with $\phi_\infty = 0$ as well. In this case $V''_\infty = V''(0) < 0$, and the linear set (49) reduces to the following asymptotic equation for $\phi(l)$:

$$\phi'' + (d_0+1)\gamma'_\infty \phi' - |V''_\infty| \phi = 0. \tag{54}$$

Its general solution is a linear combination of vanishing and growing functions:

$$\phi = A e^{-\lambda_+ l} + B e^{-\lambda_- l}, \qquad \lambda_\pm = \frac{(d_0+1)\gamma'_\infty}{2} \pm \sqrt{\frac{(d_0+1)^2 \gamma'^2_\infty}{4} + |V''_\infty|}, \qquad l \to \infty. \tag{55}$$

Requirement of regularity demands to exclude the growing solutions from the consideration. It can be done at the expense of ϕ'_0. Regular solutions terminating at a maximum of the potential can exist only at some fixed values of ϕ'_0.

I. Boundary conditions

The complete set of equations determining the structure of topological defect in the case of vector order parameter (28, 30, 31) is of the third order with respect to three unknowns γ', β', and ϕ. The simple solution is determined unanimously by the values of these three functions in any regular point. The center $l = 0$ is a singular point of the cylindrical coordinate system. The condition $\phi(0) = 0$ fulfills for both symmetries (high and broken).

β' is infinite at $l = 0$. We have to set the boundary conditions very close to the center, but not exactly at $l = 0$.

For numerical analysis it is convenient to deal with a system of four first order equations solved against the derivatives (41). The symmetry breaking potential $V(\phi)$ enters the equations (41) only via its derivative $\frac{\partial V}{\partial \phi}$. If we leave only the main terms in the boungary conditions: $U = -\frac{1}{l}$, $W = \frac{1}{l}$ at $l \to 0$, then we loose any information about the absolute value of the potential. The value $V_0 = V(0)$ appears in the next approximation. Using the expansion (14) of β' in the vicinity of the center and the equation (44), we express c via γ_0'' :

$$c = -(d_0 - 2)\gamma_0''. \tag{56}$$

To preserve the complete information about the symmetry breaking potential one has to write the boundary conditions at $l \to 0$ as follows

$$U = \tfrac{1}{3}(d_0+1)\gamma_0'' l - \tfrac{1}{l}, \quad W = \tfrac{2}{3}(d_0+1)\gamma_0'' l + \tfrac{1}{l}, \quad \phi = \phi_0' l, \quad Z = 2\phi_0'. \tag{57}$$

The values γ_0'', ϕ_0', and V_0 are not independent. They are connected with each other by (46).

J. Solutions in case $\frac{\partial V}{\partial \phi} = 0$

If the potential $V = V_0$ does not depend on ϕ, then it actually plays the role of the cosmological constant $\Lambda = \kappa^2 V_0$. The peculiarity of the vector order parameter is that the equations (41) loose the information about the potential if $\frac{\partial V}{\partial \phi} \equiv 0$. V_0 is present only in the boundary conditions (57). The equations (41) with $\frac{\partial V}{\partial \phi} \equiv 0$ and boundary conditions (57) have the following analytic solution

$$U = -\frac{\sqrt{C}}{\sinh\left(\sqrt{C}l\right)}, \quad W = \sqrt{C}\coth\left(\sqrt{C}l\right), \quad \phi(l) = \frac{2\phi_0'}{\sqrt{C}}\tanh\frac{\sqrt{C}l}{2},$$

where

$$C = 2(d_0+1)\gamma_0'' = -\frac{2(d_0+1)}{d_0}\left(2\kappa^2\phi_0'^2 + \Lambda\right). \tag{58}$$

The solution is regular if $C \geq 0$, i.e. $\Lambda \leq -2\kappa^2\phi_0'^2$. For $g_{00} = e^{2\gamma}$ and $r = e^{\beta}$ we find

$$g_{00}(l) = e^{2\gamma} = \left(\cosh\frac{\sqrt{C}l}{2}\right)^{\frac{4}{d_0+1}}, \quad r(l) = \frac{2\sinh\left(\frac{\sqrt{C}l}{2}\right)}{\sqrt{C}}\left(\cosh\frac{\sqrt{C}l}{2}\right)^{-\frac{d_0-1}{d_0+1}}.$$

The slope ϕ_0' remains arbitrary. If $\phi_0' = 0$ this solution reduces to the one found earlier (see [1] and [3]) for the special case $\phi \equiv 0$. The point is that the Einstein equations with a negative cosmological constant have a nontrivial solution (with a nonzero order parameter) even without a symmetry breaking potential.

The necessary condition of regular solutions with broken symmetry is the existence of extremum points of $V(\phi)$, where $\frac{\partial V}{\partial \phi} = 0$. In case $V = const$ the condition $\frac{\partial V}{\partial \phi} = 0$ is fulfilled identically, and formally the order parameter ϕ can tend to any ϕ_∞ as $l \to \infty$. The displayed above analytical solution shows that the existence of a negative cosmological constant is sufficient for the symmetry breaking of a uniform plain bulk.

The special case $C = +0$ in (58) when $\gamma_0'' = 0$ and

$$\phi_0' = \pm\sqrt{-\frac{\Lambda}{2\kappa^2}} \tag{59}$$

corresponds to the plain bulk $g_{00}(l) = 1$, and $r(l) = l$.

K. Weak curvature of space-time

The limit $\kappa^2 \to 0$ is the transition to a flat space-time. Functions β' and γ' reduce to $\beta' = l^{-1}$, $\gamma' = 0$. The field equation (22) reduces to (24), which is the usual equation for the order parameter in cylindrical coordinates in a flat space-time. The symmetry breaking potential V is a function of ϕ^2, so $\frac{\partial V}{\partial \phi} \sim \phi$, and (24) has a trivial solution $\phi = 0$ corresponding to the symmetric (not broken) state. The nontrivial solutions, starting with $\phi(0) = 0$, $\phi'(0) \neq 0$ and terminating with $\phi = \phi_m$ at an extremum of the potential $\left(\frac{\partial V(\phi_m)}{\partial \phi} = 0\right)$, describe the states of broken symmetry. Equation (24) is nonleniar. However, depending on the form of the potential $V(l)$ it can also have a sequence of nontrivial solutions $\phi_n(l)$, $n = 0, 1, 2, ...$, with zero boundary conditions $\phi(0) = \phi(\infty) = 0$ on both ends. The discrete sequence of derivatives $\lambda_n = \phi_n'(0)$ forms the eigenvalues for the eigenfunctions $\phi_n(l)$. Functions $\phi_n(l)$ change sign n times. The nontrivial solutions of the field equation with $\phi'(0)$ within the interval $(\lambda_n, \lambda_{n+1})$ change sign $n+1$ times.

The principle difference between the equations (22) and (24) is that the coefficient $(d_0\gamma' + \beta')$ at ϕ' in curved space-time doesn't vanish at $l \to \infty$. If $\phi = \phi_m$ is a minimum of $V(\phi)$ then $V''(\phi_m) > 0$, and the linearized field equation (24) in case of flat space-time at

$l \to \infty$ reduces to

$$\phi'' + V''(\phi_m)(\phi - \phi_m) = 0$$

and describes non-vanishing oscillations. In curved space-time the oscillations vanish at $l \to \infty$ in accordance with (51).

Further detailed analysis is done with the aid of numerical integration.

IV. NUMERICAL ANALYSIS

A. Regular solutions in the space of parameters

The numerical integration of equations (41) is performed for the "Mexicam hat" potential taken in the same form as in [1]:

$$V = \frac{\lambda\eta^4}{4}\left[\varepsilon + \left(1 - \frac{\phi^2}{\eta^2}\right)^2\right] \tag{60}$$

The potential (60) has three extremum points – a maximum at $\phi = 0$, and two minima at $\phi = \pm\eta$. At the limiting values of the order parameter

$$\begin{aligned} V'_\infty &= 0, \quad V''_\infty = 2\eta^2, \qquad \phi_\infty = \pm\eta \\ V'_\infty &= 0, \quad V''_\infty = -\eta^2, \qquad \phi_\infty = 0. \end{aligned}$$

The dimensionless parameter ε moves the "Mexican hat" up and down. It is equivalent to adding a cosmological constant. The energy of spontaneous symmetry breaking is characterized by $\eta^{2/(D-2)}$, and

$$a = \frac{1}{\sqrt{\lambda}\eta} \tag{61}$$

determines, as usual, the length scale. In most cases a is associated with the core radius of a topological defect. Without loss of generality we set $a = 1$ in computations. The strength of gravitational field is characterized by the dimensionless parameter

$$\Gamma = \kappa^2\eta^2. \tag{62}$$

In the case of vector order parameter the state of broken symmetry is controled by four parameters d_0, ε, Γ, and ϕ'_0. The main difference is that in the scalar multiplet case regular

configurations with given d_0, ε, and Γ existed only for a fixed value of ϕ_0'. Now the regular configurations with given d_0, ε, and Γ exist within some interval $0 < \phi_0' \leq \phi_{0max}'$ with the upper boundary ϕ_{0max}' depending on d_0, ε, and Γ. This additional parametric freedom allows to forget about the so called "fine tuning" of the physical parameters.

For visual demonstration it is worth to fix $d_0 = 4$ and one of the three other parameters. Then the area of existence of regular solutions can be presented as a map in the plane of two remaining parameters.

Fig 1. shows the area of regular configurations in the plane (ε, ϕ_0') for $d_0 = 4$ and $\Gamma = 1$. Depending on the values of ε and ϕ_0' the order parameter $\phi(l)$ tends to $+\eta$, 0, or $-\eta$ as $l \to \infty$. The sequence of curves $f_n(\varepsilon)$ in Fig. 1 are those where $\phi(l) \to 0$ as $l \to \infty$. They separate the areas with different signs of ϕ_∞. Below the first curve $f_1(\varepsilon)$ from the bottom, where $0 < \phi_0' < f_1(\varepsilon)$, the order parameter $\phi(l)$ doesn't change sign. Between $f_1(\varepsilon) < \phi_0' < f_2(\varepsilon)$ it changes the sign once. In the area $f_2(\varepsilon) < \phi_0' < f_3(\varepsilon)$ it changes the sign twice, and so on. The curves $f_n(\varepsilon)$ quickly condense to the upper red curve $f_\infty(\varepsilon)$ as $n \to \infty$. $f_\infty(\varepsilon)$ is the upper boundary of existence of regular solutions (in the particular case $d_0 = 4$ and $\Gamma = 1$).

The curves in Fig. 1 are those where

$$\phi_\infty(\phi_0', \varepsilon, d_0 = 4, \Gamma = 1) = 0. \tag{63}$$

Similar curves can be shown for fixed ϕ_0' in the plane (ε, Γ). For instance, the dash line in Fig. 2 is the first one of the curves $\phi_\infty\left(\phi_0' = \pm\sqrt{-\frac{\varepsilon+1}{8}}, \varepsilon, d_0 = 4, \Gamma\right) = 0$, where the order parameter tends to zero at $l \to \infty$. The value $\phi_0' = \pm\sqrt{-\frac{\varepsilon+1}{8}}$ (59) corresponds to $\gamma_0'' = 0$ in (46). It is the case $C = 0$ in (58), so that the symmetry breaking of the plain bulk is caused completely by the potential $V(\phi)$, and not by the cosmological constant. To the right of the dash line $\phi(l)$ does not change the sign.

For the potential (60) the boundary line (53) between oscillating and smooth $\phi(l)$ is

$$-\varepsilon_b = 16\frac{(1+G)^2}{G}, \quad G = \frac{d_0+1}{d_0\Gamma}. \tag{64}$$

It is presented in Fig. 2 (solid line). Below the solid line the order parameter $\phi(l)$ tends to its limiting value ϕ_∞ with damping oscillations (see Fig. 3), and above this curve – without oscillations, see Fig. 4. The curves in Fig. 3 correspond to the close vicinity of the

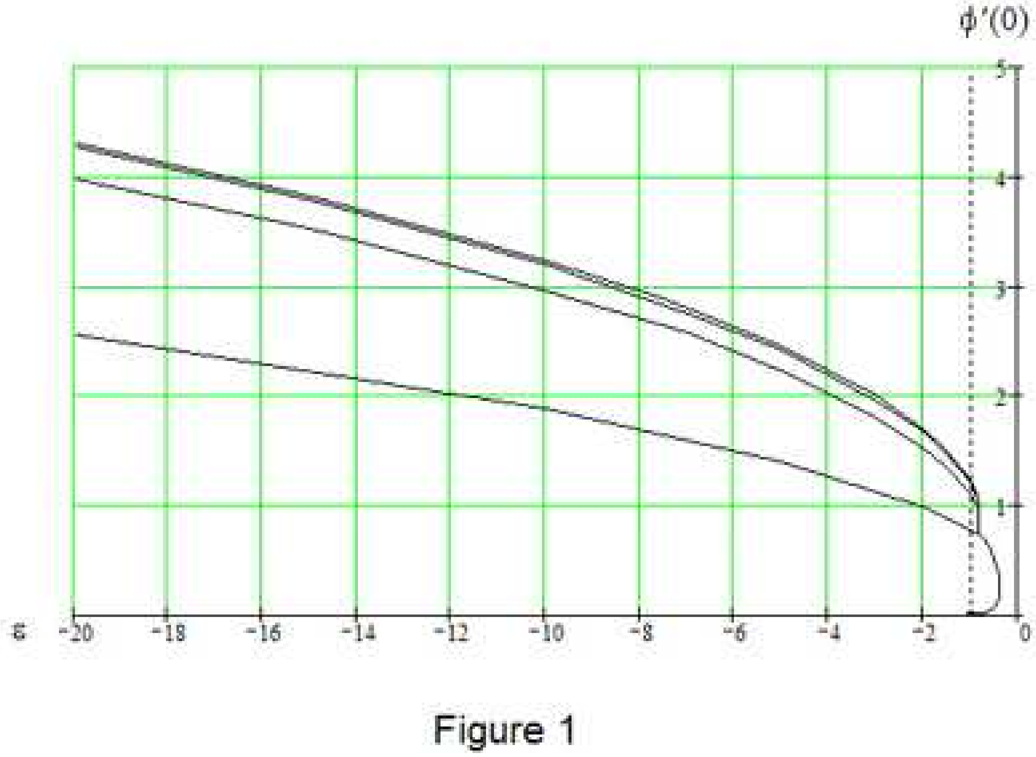

FIG. 1: The area of regular configurations in the plane (ε, ϕ_0') for $d_0 = 4$ and $\Gamma = 1$. The upper curve is the boundary of existence of regular solutions. Other curves separate the regions with different signs of ϕ_∞. They quickly condense to the upper curve. Below the lower curve $\phi(l)$ does not change sign.

lower black point on the dash curve in Fig. 2, the curves in Fig. 4 – to the vicinity of the upper black point.

B. Neutral quantum particle in the space-time with metric (9)

A neutral spinless quantum particle is described by a scalar wave function χ with the Lagrangian

$$L_\chi = \frac{1}{2} g^{AB} \chi^*_{,B} \chi_{,A} - \frac{1}{2} m_0^2 \chi^* \chi. \tag{65}$$

In the uniform bulk (while the symmetry is not broken) it is a free particle in the D-dimensional space-time with mass m_0 and spin zero. In the broken symmetry space-time

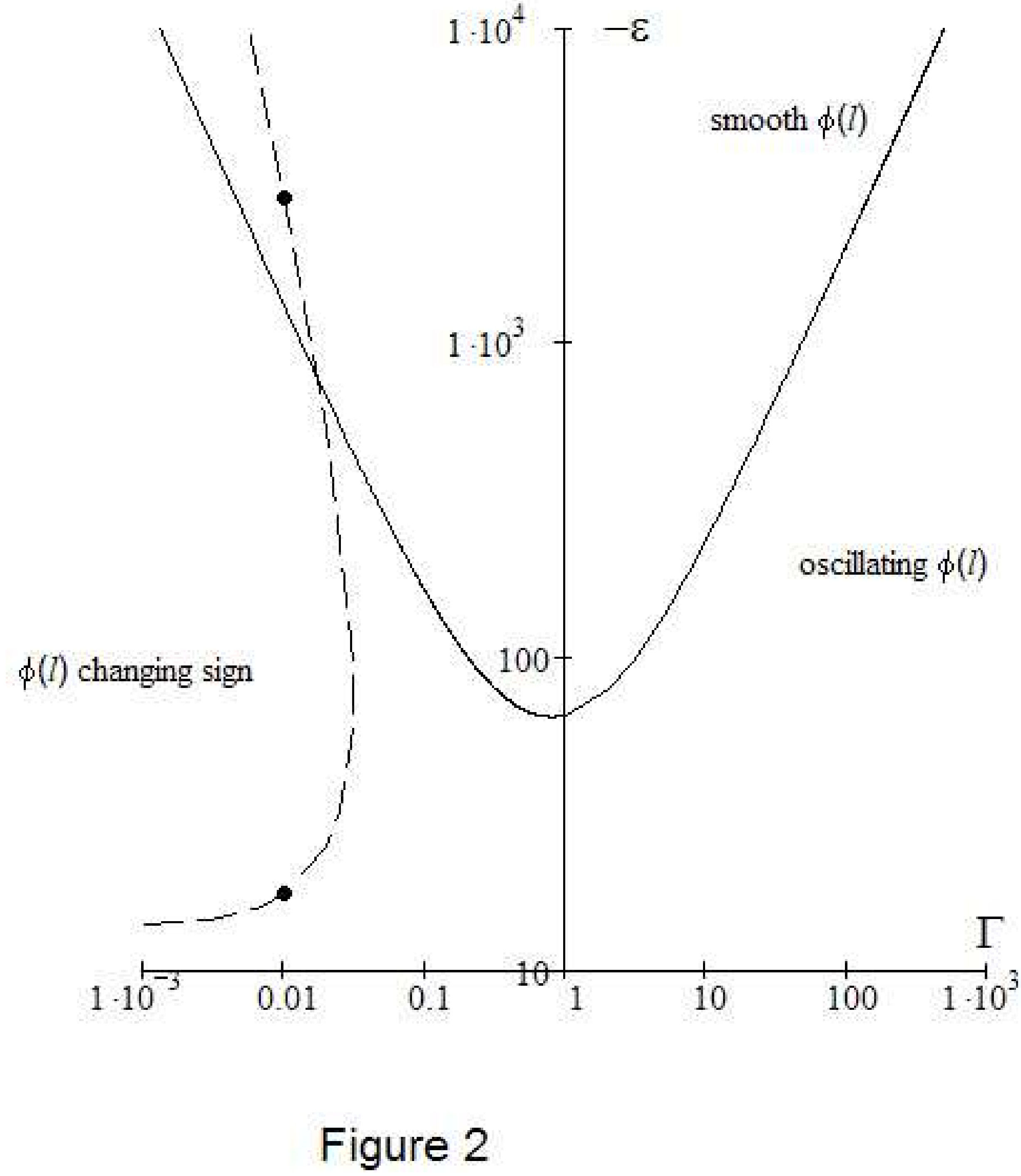

FIG. 2: Map of regular solutions in the plane $(\Gamma, -\varepsilon)$ for $\phi_0' = \pm\sqrt{-\frac{\varepsilon+1}{8}}, d_0 = 4$. The red curve separates the regions of smooth (above) and oscillating (below) behavior of the order parameter at $l \to \infty$. To the left of the blue curve the order parameter changes sign.

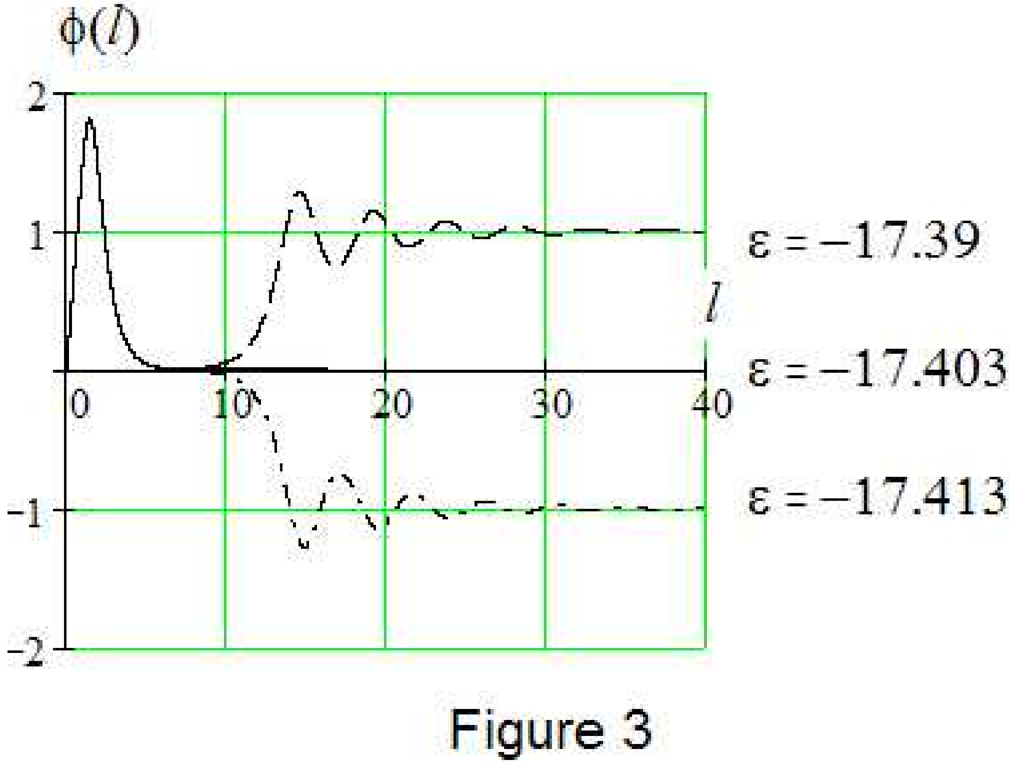

FIG. 3: Oscillating solutions in the close vicinity of the lower red point on the blue curve in Fig. 2.

with metric (9) it satisfies the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{AB}\chi_{,A}\right)_{,B} + m_0^2\chi = 0. \tag{66}$$

All coordinates except $x^{d_0} = l$ are cyclic variables, and the conjugate momenta are quantum numbers. The wave function in a quantum state is

$$\chi\left(x^A\right) = X\left(l\right)\exp\left(-ip_\mu x^\mu + in\varphi\right), \tag{67}$$

where $p_\mu = (E, \mathbf{p})$ is the d_0-momentum within the brane, and n is the integer angular momentum conjugate to the circular extradimensional coordinate φ. $X\left(l\right)$ satisfies the equation[1]

$$X'' + WX' + \left(p^2e^{-2\gamma} - n^2e^{-2\beta} - m_0^2\right)X = 0. \tag{68}$$

The eigenvalues of $p^2 = E^2 - \mathbf{p}^2$ compose the spectrum of squared masses, as observed in the brane. Quantum number n is the integer proper angular momentum of the particle. From the point of view of the observer in the brane it is the internal momentum, identical to the spin of the particle.

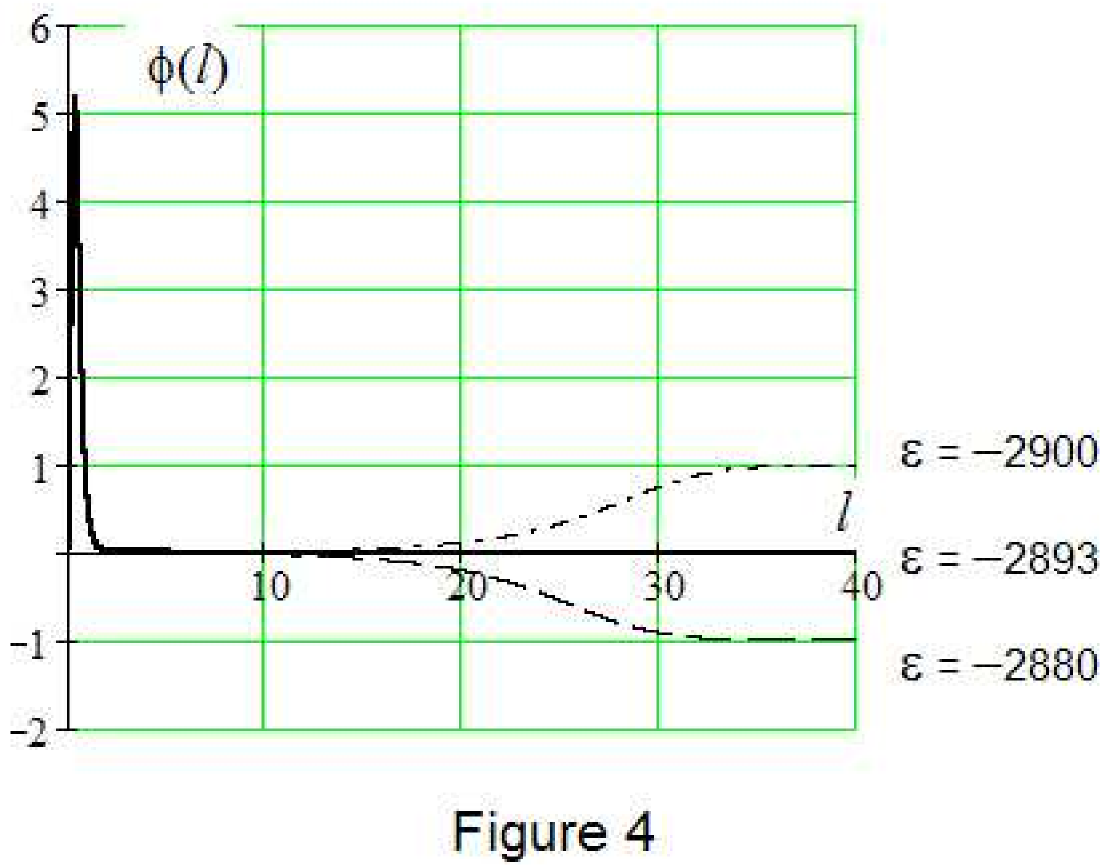

FIG. 4: Smooth solutions in the close vicinity of the upper red point on the blue curve in Fig. 2.

The equation (68) takes the form of the Schrodinger equation

$$y_{xx} + \left[p^2 - V_g\left(x\right)\right] y = 0 \tag{69}$$

after the substitution

$dl = e^{\gamma} dx, \qquad X\left(l\right) = y\left(x\right)/\sqrt{f\left(x\right)}, \qquad f\left(x\right) = \exp\left\{-\frac{1}{2}\left[\left(d_0 - 1\right)\gamma + \beta\right]\right\}.$

The gravitational potential

$$V_g\left(x\right) = e^{2\gamma}\left(e^{-2\beta} n^2 + m_0^2\right) + \frac{1}{2}\frac{1}{\sqrt{f}}\frac{d}{dx}\left(\frac{1}{f^{1/2}}\frac{df}{dx}\right) \tag{70}$$

determines the trapping properties of particles to the brane. In terms of U, W and ϕ (33) the dependence of the gravitational potential (70) on the distance l is

$$V\left(l\right) = e^{2\gamma}\left(e^{-2\beta} n^2 + m_0^2\right) + \frac{e^{2\gamma}}{4}\frac{\left(d_0 W - U\right)\left(U + \left(d_0 + 2\right)W\right)}{\left(d_0 + 1\right)^2} + \frac{e^{2\gamma}}{2}\left[\kappa^2 \frac{\partial V}{\partial \phi}\phi + \frac{U\left(W - d_0 U\right)}{d_0 + 1}\right]. \tag{71}$$

C. Oscillations

In terms of (64) the eigenvalues (52) are

$$\lambda_{\pm} = -\sqrt{-\frac{\varepsilon}{8\,(G+1)}}\left[1 \pm \sqrt{1 + \frac{16}{\varepsilon G}\,(G+1)^2}\right]. \tag{72}$$

The less is $|\varepsilon|$ the more oscillations display themselves. In the limiting cases of small and large Γ the frequencies of oscillations

$$|Im\lambda| = \begin{cases} \sqrt{2}, & \Gamma \to 0, \\ \sqrt{2\left(1 + \frac{1}{d_0}\right)\Gamma}, & \Gamma \to \infty \end{cases}$$

do not depend on ε as $l \to \infty$.

The oscillations of the order parameter $\phi(l)$, see Fig.5, induce the oscillations of the gravitational potential (70). At $|\varepsilon| \sim 1$ and $\Gamma \gg 1$ the gravitational potential has many points of minimum, see Fig.6.

The length scale a (61) remains an arbitrary parameter of the theory. The physical interpretation is different in the limiting cases of large and small a. If a is extremely large, each minimum of the potential $\gamma(l)$ forms its own brane. If the potential barrier is high, the branes are separated from one another.

In the opposite limit, when the scale length a is extremely small, all points of minimum are located within one common brane, and in the spirit of Kalutza-Kline the points of minimum are beyond the resolution of modern devices.

Low energy particles can be trapped by the points of minimum of the potential (70). Identical in the bulk neutral spin-less particles, being trapped in the different minimum points, acquire different masses and angular momenta. If the scale length a is extremely small, then for the observer within the brane they appear as different particles with integer spins.

Most elementary particles have half-integer spins. The simple case of spontaneous symmetry breaking, considered above, can not connect the origin of half-integer spins with extra-dimensional angular momenta.

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[1] K.A.Bronnikov and B.E.Meierovich. Zh.Eksp. Teor. Fiz. Vol. **133**, No. 2, pp. 293-312 (2008).

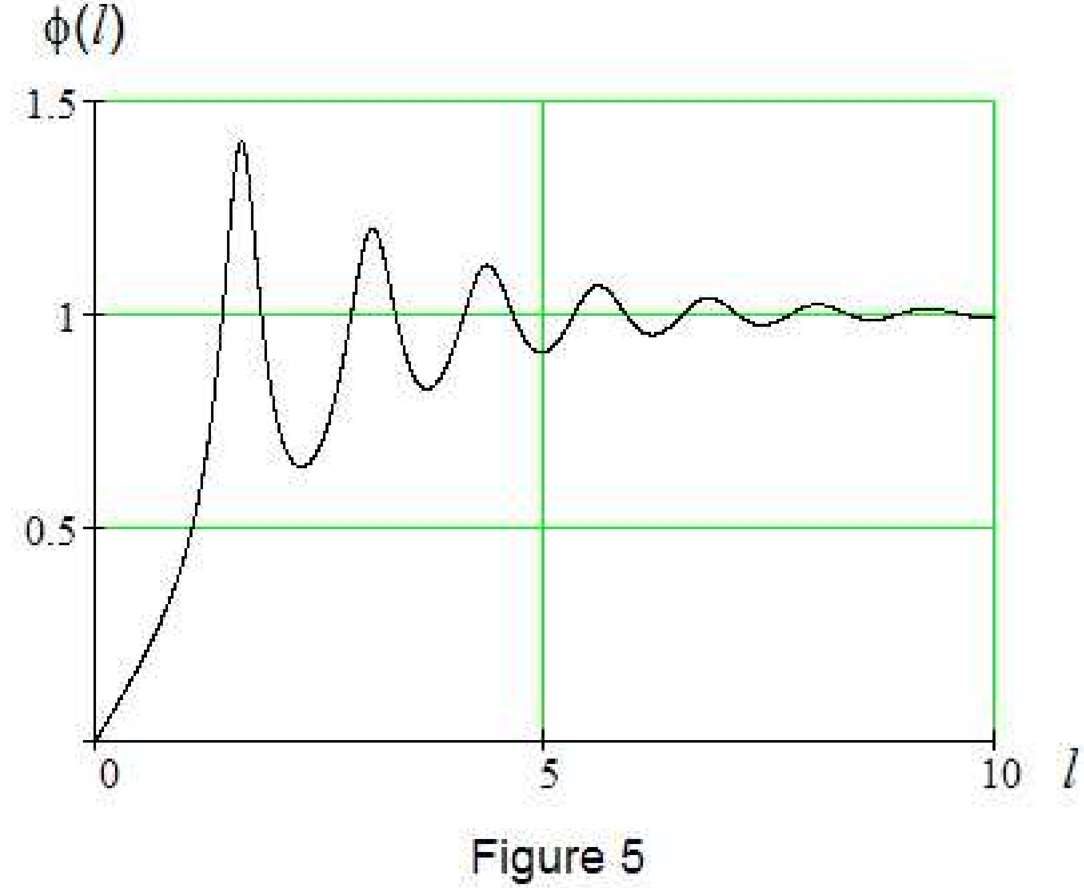

FIG. 5: A solution with oscillating order parameter $\phi(l)$. Here $d_0 = 4, \varepsilon = -2, \Gamma = 10, \phi_0' = \sqrt{-\frac{\varepsilon+1}{8}}$.

[2] L.D.Landau and E.M.Lifshits. Field Theory. “Nauka”, Moscow, 1973.

[3] J.M.Cline, J.Descheneau, M.Giovannini, and J.Vinet. E-print archives, hep-th/0304147v2.

[4] It differs from (94.4) in [2] because the Lagrangian is considered there as a function of g^{IK} and $\frac{\partial g^{IK}}{\partial x^L}$. Here and below $\sqrt{-g}$ stands for $\sqrt{(-1)^{D-1}g}$.

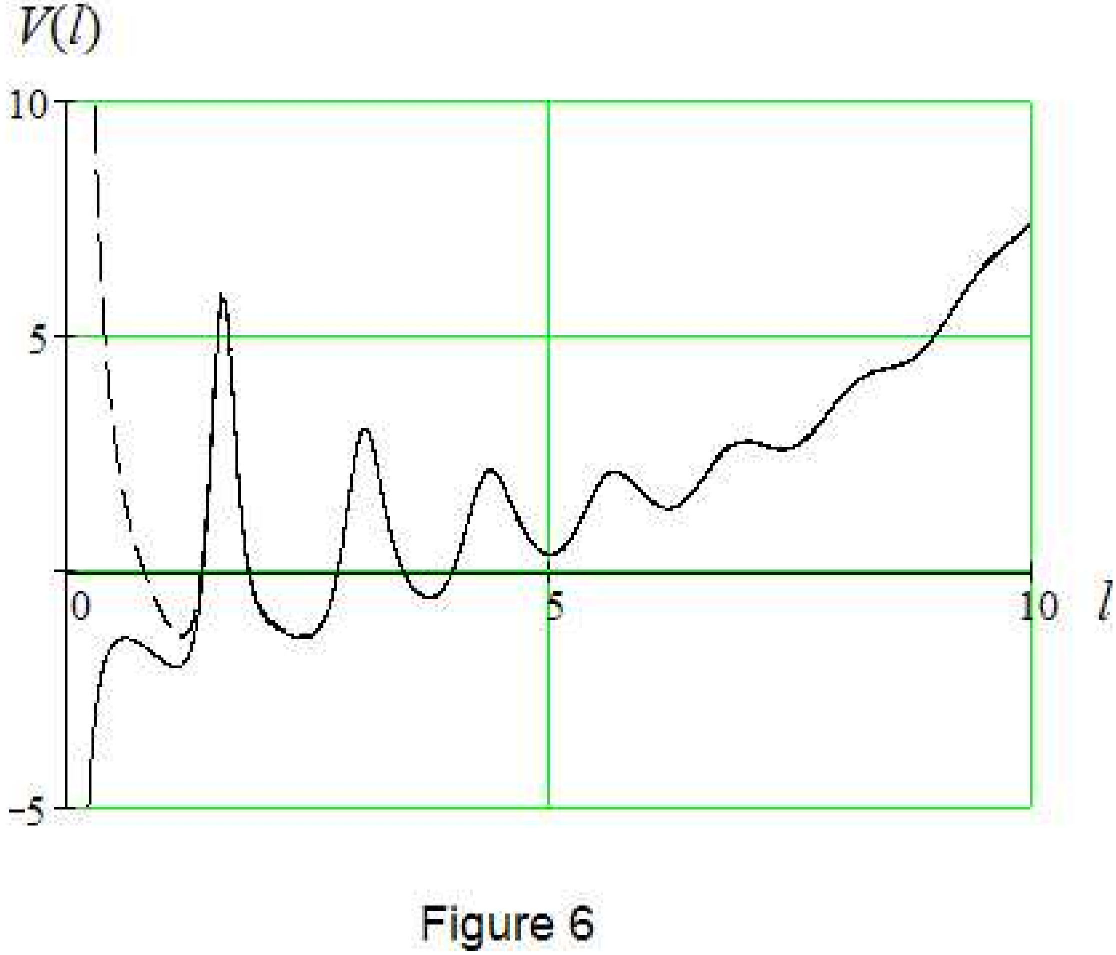

FIG. 6: Gravitational potential $V_g(l)$ for the same set of the parameters as in Fig.5, $d_0 = 4, \varepsilon = -2, \Gamma = 10, \phi_0' = \sqrt{-\frac{\varepsilon+1}{8}}$. The initial mass of a test particle is set $m_0 = 0$. The red curve corresponds to the angular momentum $n = 0$, and the dashed blue one – to $n = 1$.